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Mathematical Habits of the Mind for Preservice Teachers

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ABSTRACT

This article provides an overview of mathematical habits of the mind and discusses how the concept relates to Polya's problem solving principles as well as exemplification. Specific problems are discussed as a means to assist preservice elementary school teachers' in their development of mathematical habits of the mind. Without a technique to begin solving these rich problems, preservice teachers may have difficulty getting started. The process of preservice teachers outlining their thinking as they progress through Polya's process is discussed. Students' reflections from this technique are discussed to explore the outcomes that may be expected from establishing an environment where students are encouraged to develop mathematical habits of the mind.

Promoting a Mathematical Habit of the Mind in Preservice Teachers

An approach to promoting mathematical habits of the mind is to develop a culture of sound mathematical questioning in present and future teachers that will allow them to challenge themselves and their students at an appropriate level. An overview of mathematical habits of the mind will be provided followed by problems that may help preservice teachers think about mathematics differently. Finally, a specific strategy will be discussed to help students think about how to get started with rich mathematical problems. Preservice teachers' reflections upon their experiences provide a valuable lesson regarding how this technique may help preservice teachers (and their future students) develop mathematical habits of the mind. While we believe that this approach is appropriate at all levels, in order to narrow the scope of this article, the focus of this paper will be on infusing the idea of a mathematical habit of the mind in the preparation of future elementary and middle school teachers.

Overview of Mathematical Habits of the Mind

To help their own students be more creative, teachers need to work with their students to: 1) explore mathematical ideas; 2) formulate questions; 3) construct examples (Watson & Mason, 2005); 4) identify problem solving approaches that are useful in large classes of problems; 5) ask themselves whether there is "something more" (a generalization) in the mathematics on which they are working; and 6) reflect on their answer to see whether they have made an error (Jacobbe, 2007). These traits can be called "mathematical habits of the mind" which is a term that is featured prominently in an important teacher education report from the Mathematical Association of America (CBMS, 2001). Furthermore, the process standards set forth in the *Principles and*

Standards for School Mathematics (NCTM, 2000) include problem solving, communication, reasoning and proof, making connections and representations. These processes can be described as mathematical habits of the mind.

Problem solving is at the heart of mathematical habits of the mind. In *Benchmarks for Science Literacy*, The American Association for the Advancement of Science (AAAS) indicated that "preparing students to become effective problem solvers, alone in concert and with others, is a major purpose of schooling (AAAS, 1993, 282)." One of the first mathematicians to clearly articlulate the process of problem solving was George Polya. Polya's (1956, 1973) foundational writing on problem solving involves the creation of mathematical habits of the mind, although he does not use the phrase. The Polya method consists of four steps: understanding the problem, devising a plan, carrying out the plan, and looking back (Polya, 1973, xvi-xvii).

It is the "looking back" step that is very much the heart of mathematical habits of the mind as described above. "By looking back at the completed solution, by reconsidering and re-examing the result and the path that led to it, they [students] could consolidate their knowledge and develop their ability to solve problems (Polya, 1973, pp. 14-15)." In *Benchmarks for Science Literacy*, the AAAS emphasized the importance of reflection as well.

> Learning to solve problems in a variety of subject-matter contexts, if supplemented on occasion by explicit reflection on that experience, may result in the development of a generalized problem-solving ability that can be applied in new contexts; such transfer is unlikely to happen if either varied problem-solving experiences or reflection on problem solving is missing. (AAAS, 1993, 282)

Jacobbe (2007) used the looking back or reflection step as a way to help structure students' thinking in order to overcome translation difficulties. Reflections from students' experiences with this approach are shared later in the article as a means to articulate a method to help students develop mathematical habits of the mind.

Herbert Clemens (1991) was one of the first mathematicians to advocate that, in order for teachers to be successful, they must create an environment that promotes mathematical thinking. Mathematics is not about all the rules and procedures that must be memorized and regurgitated by students; rather, it is about developing a process for approaching various problems. Clemens (1991) argues that two equally successful teachers could choose entirely different sets of problems to center their curriculum around as long as they have “mathematical integrity and equal relevance to quantitative experiences and questions that are natural to human beings (Clemens, 1991, p. 87).”

A review of the literature has revealed no studies on this subject except for those which are revolve around exemplification (See Watson & Mason, 2005). There have been discussion pieces centered at the high school level (See Cuocco, 1996, 2001). The idea of “exemplification in mathematics” fits in well with the notion of mathematical habits of the mind. Watson & Mason (2005) carefully explain the framework, application, history, and research base of this approach.

> Example creation can provide an arena not only for practice but also for conceptual learning…With every construction there is also a question: How much choice do I have? Mathematicians typically pursue this by then asking whether they can characterize all such examples in some way. (Watson & Mason, 2005, p. 13)

This strategy consists of having learners construct their own examples as a way to give insights into more advanced ideas. “No matter how profoundly one thinks one understands, it is always possible to probe more deeply and to discover more connections and complexities (Watson & Mason, 2005, p. xiv).”

In order to foster mathematical habits of the mind, it is important to pose questions that allow for multiple solution paths. The following is an example of a high school mathematics problem posed to mathematicians by Watson and Mason:

> We have decided to call the horizontal distance between neighboring roots *f* a polynomial function the inter-rootal distance. Imagine a quadratic equation with two real roots. What families of quadratic curves have the same inter-rootal distance? (Watson & Mason, 2005, 40)

In their discussion of this example Watson and Mason revealed the variety of methods that the mathematicians came up with. The authors relate the solutions to the background and area of specialization of the mathematicians. One used dynamic software packages and another found one member of the family and then, with some thought, used translation and stretching the y-axis. Two of the conclusions of Watson and Mason is that "example creation is individual (2005, 41)." and, "Being a mathematician includes being able to question one's first response and seek alternatives, searching algebraically, geometrically, and in other ways to find as many examples as one can (2005, 42)."

Mathematical habits of the mind, as outlined in this paper, do not require a deep mathematical background, only an inquiring mind that will focus on thinking in a mathematically creative way. Nor does it have to be something new to the world or even ideas that are novel or innovative — rather focusing on the creation of mathematical habits of the mind is a way to structure, infuse, and talk about some aspects of mathematics.

Examples of Mathematical Habits of the Mind at the Elementary and Middle School Level

This section presents problems that can be used to promote mathematical habits of the mind among preservice teachers. It is important for preservice teachers to experience these types of problems so that they may know how to create an environment that pushes their future students' thinking.

The following example was given in a content course on problem solving for elementary and middle school teachers. After proving that the sum of even numbers is even, the sum of odd numbers is even, etc., the class was asked to show that the product of even numbers is even. At the end of that in-class group exercise, the instructor asked the students to think back on what happened and whether there was anything more going on. After more conversations in their groups, and a series of hints from the instructor to some of the groups,the future teachers realized that the product was actually divisible by four, not just two and that, as one of them said, "the result is free" meaning that the same proof worked for the stronger result. We have given similar "is there more?" questions, such as ones concerning the square of an odd number, in the content math course required of all future elementary school teachers.

In the same course, students were also asked to show that a certain polynomial expression in the integer *n* is divisible by 4, 8, 16 or 32. An example of this type of problem is as follows:

> Let *n* be an odd integer. Show that 16 divides $(n^2+4n-1)(n^2+4n+3)$. Does 32 divide the expression? (Millman, 2007)

Average students can do the first question and can show that 32 also divides the expression in special cases. The top 20% of the class was able to write out a proof of divisibility by 32 by using algebra. The division by 16 is "low hanging fruit" (just follow the algebra) but division by 32 requires an insight (that, after simplifying, there is a product of 16 and two consecutive integers). Said another way, insights and a conceptual understanding are very much examples of mathematical habits of the mind, whereas routine computation is not.

Fermi Problems, which are open-ended or vaguely worded problems, would also foster the development of mathematical habits of the mind since they require students to examine how they would model a situation (see Peter-Koop, 2005). Preservice elementary school teachers in a mathematical content course have found these types of problems quite intriguing and are delighted to see what German fourth grade students do on the problems.

Another example of a problem that may help preservice teachers develop mathematical habits of the mind can be shown by three exercises which present the same idea in three different ways as follows:

a. Find all pairs of prime numbers that sum to 313.
b. If k is an odd number, what possible pairs of prime numbers can sum to k?
c. If k is an odd number, show that there are either no prime numbers which sum to k or only one pair of primes that sum to k.

Future elementary school teachers have been asked to compare these questions to demonstrate an example of insights along the lines of mathematical habits of the mind. Although the first question may seem easier to get started with to a preservice elementary school teacher, the second two questions are actually easier to answer. The first question requires the insight that the real point is that 313 is odd. While these are phrased as questions (or prompts), not examples, the analogies between this type of exercise and the approach of Watson & Mason are clear.

Two other examples, which come from a textbook designed for future elementary school teachers, are provided below. The first asks preservice teachers to recognize what it means for a statement to always be true and see that algebra can be used for this purpose. The notion of algebra as a way of proving conjectures is a mathematical habit of the mind. The first problem is:

> A student has noticed that every squared natural number is either a multiple of 4 or one larger than a multiple of 4, but isn't sure if this is always true. Use algebraic reasoning to explain to the student why this is always true. (DeTemple, Millman, & Long, 2008)

Another example of a problem that promotes the development of mathematical habits of the mind follows from a paragraph in the text, which is neither an example nor an exercise, and asks preservice teachers to think about the relationships between two different concepts. In this case, the ideas underneath the prose are that we can derive a formula by relating two quantities which seem unrelated and that areas or volumes can be approximated by very small areas or volumes.

> We will now derive a formula for the surface area of a sphere of radius r by making use of the formula that we already know for the volume of a sphere. This derivation is done through reasoning, rather than by formal proof, as is the approach of this chapter. The key concept will be to relate the volume of the sphere to the volume of pyramid-like solids whose base is a small piece of the surface. We first divide the sphere's surface into many tiny regions of area…(DeTemple, Millman, & Long, 2008)

The problems discussed above are interesting and may help preservice teachers; however many students struggle with how to begin when presented with problems where the solution methods are not straightforward. One technique that can be employed to help preservice teachers think about mathematics differently is to require students to outline their thinking as they progress through each of Polya's steps in the problem solving process. Particular attention should be paid to students' outlining their thinking at the "looking back" stage in the process. As described above, the "looking back" stage is at the heart of mathematical habits of the mind. Jacobbe (2007) showed how this technique can greatly improve students' performance on translation tasks (incorrect response rate reduced from 79 to 14 percent). As exhibited through students' reflections, this technique may also empower preservice teachers to think about mathematics differently.

Students Reactions to Progressing through Polya's Steps

The process of outlining their thinking at each of Polya's steps required the students to really think about what they were doing. Many students struggled with where to begin; however Polya's first step "understanding the problem" often helped them "devise a plan." This activity of progressing through the steps has the potential to impact preservice elementary school teachers' future approaches to teaching mathematics in the classroom. The impact of progressing through the steps is evident in the following students' responses to a question regarding the impact of progressing through Polya's steps as they solved several problems.

> Math has always been a hard subject for me. However, I have the most difficulty with word problems. With these steps, I can figure out how to do the problems effectively.

Student Response 1

> This process would help me solve other problems that I would have difficulty with. By following these steps one is forced to analyze and outline each area of a problem. This not only breaks the problem down to seem simpler, but enables the problem solver to visually catch his/her mistakes or point out where he/she may have gotten stuck. It also makes it easier to go back and check final answers.

Student Response 2

> The process would help with any problem that would otherwise be difficult.

It makes it easy to see what you need to do when you follow each step carefully. You can see the problems in different ways than you may have thought of. The problems may be simpler than you originally realized. The steps can help you see everything in the easiest way possible.

Student Response 3

Initially, I felt that progressing through each step of the word problems was time consuming and repetitive. However, I realized that learning and carrying out these steps in each of the problems is important for elementary school students as well as teachers. Each step in the problem solving process is important to understanding the problem fully. When a student comes in contact with a difficult problem, it is necessary for him/her to complete the steps to come out with a correct answer and a full understanding. Also, the problem solving process will help teachers be able to explain their answer more effectively. I have come to realize that even though completing all the steps may be time consuming, it is an effective way to solve problems.

Student Response 4

The impact of having to progress through each step in Polya's process is most evident in Student Response 4. This particular student realized the importance of students and teachers outlining their thinking as they work through various problems. This technique may provide preservice teachers with a means to get started on many of the problems discussed above. Once they are free to think about the mathematics involved, they may begin to develop the mathematical habits of the mind discussed in this paper.

End Remarks

The importance of developing mathematical habits of the minds is exemplified by the contributions of Fields Medalist, Andrew Wiles. His early acquaintance with Fermat's Last Theorem and his internal desire to seek a solution sparked a career that led to solving a problem that had remained unsolved for hundreds of years. According to Wiles, "The definition of a good mathematical problem is the mathematics it generates rather than the problem itself (Wiles Interview, 2007)." Rich problems along with a technique to get them started will help preservice teachers and their future students develop mathematical habits of the mind. These habits have the power of launching internal inquisitiveness that may lead to creative mathematical discoveries.

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